On one-dimensional stochastic differential equations involving the maximum process

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Abstract. We prove existence and pathwise uniqueness results for four different types of stochastic differential equations (SDEs) perturbed by the past maximum process and/or the local time at zero. Along the first three studies, the coefficients are no longer Lipschitz. The first type is the equation

$$X_t = \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds + \alpha \max_{0\le s\le t} X_s.$$

The second type is the equation

$$\begin{cases} X_t = \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds + \alpha \max\limits_{0\le s\le t} X_s \ + L_t^0, \\ X_t \ge 0, \ \forall t \ge 0. \end{cases}$$

The third type is the equation

$$X_t = x + W_t + \int_0^t b(X_s, \max_{0\le u\le s} X_u)ds.$$

We end the paper by establishing the existence of strong solution and pathwise uniqueness, under Lipschitz condition, for the SDE

$$X_t = \xi + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds + \alpha \max_{0\le s\le t} X_s + \beta \min_{0\le s\le t} X_s.$$

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1 Introduction

One-dimensional stochastic differential equations (SDEs) involving the past maximum process and/or the past minimum process and/or the local time process has attracted several authors (see for instance [7], [5], [3], [4], [16]).

In this paper, we will study four classes of these SDEs, three with singular coefficients and the other with Lipschitz one. In fact, we are concerned with the existence and pathwise uniqueness of solutions. Because of the lack of regularity of those latter, the usual fixed point approach cannot be applied. So our method is based on uniqueness in law and basic results about local times, namely the Tanaka's formula. We do not require strong regularity assumption on the drift coefficients. Besides, we allow the diffusion coefficient be discontinuous if it is strictly positive.

Let Ω be the set of continuous functions from R^+ into R, P the Wiener measure on Ω, $(W_t)_{t\geq 0}$ the process of coordinate maps from Ω into R, $\mathcal{F} = \sigma\{W_t,\, t \geq 0\}$, $(\mathcal{F}_t)_{t\geq 0}$ the completion of the natural filtration of W with the P-null sets of $\mathcal{F}$. Therefore $(W_t)_{\geq 0}$ is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

In Section 2, we establish both the existence of a strong solution and the pathwise uniqueness for the SDE

$$X_t = \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds + \alpha \max_{0\leq s\leq t} X_s. \tag{1}$$

In Section 3, we study the perturbed SDEs with reflecting boundary

$$\begin{cases} X_t = \displaystyle\int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds + \alpha \max_{0\leq s\leq t} X_s \ + L_t^0, \\ X_t \geq 0, \, \forall t \geq 0. \end{cases} \tag{2}$$

Actually to equation (1) we add a reflecting process which is the local time $(L_t^0)_{t\geq 0}$ of the process X at 0. Its role is to push upward the process X in order to keep it above 0, i.e. to have the condition $X \geq 0$ satisfied.
Section 4 is devoted to the study of SDEs with a drift containing the maximum process

$$X_t = x + W_t + \int_0^t b(X_s, \max_{0\leq u\leq s} X_u) ds. \tag{3}$$

Once more we show existence and uniqueness of the solution $(X_t)_{t\geq 0}$. Note that $(X_t)_{t\geq 0}$ is not a Markov process but $(X_t, \max_{0\leq s\leq t} X_s)_{t\geq 0}$ is a Markov process.

In Section 5 we deal with the doubly perturbed SDEs

$$X_t = \xi + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds + \alpha \max_{0\leq s\leq t} X_s + \beta \min_{0\leq s\leq t} X_s. \tag{4}$$

Under Lipschitz condition on the coefficients and an assumption on the parameters α, β, we prove the strong existence and pathwise uniqueness property for (4). We conclude this paper by a discussion on some further extensions of our results. $\diamond$

Throughout this paper, solutions to each stochastic differential equation under consideration should be understood as continuous processes. Besides let us recall the following definitions: A *strong solution* for (1) (resp. (2); resp. (3)) on the probability space $(\Omega, \mathcal{F}, P)$ endowed with the Brownian motion W is a continuous process $(X_t)_{t\geq 0}$ adapted w.r.t. the natural filtration of W and which satisfies (1) (resp. (2); resp. (3)). A *weak solution* for (1) (resp. (2); resp. (3)) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a couple of processes (X, W) on that space such that X is adapted with respect to $(\mathcal{F}_t)_{t\geq 0}$, W is a Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$ and finally the couple (X, W) satisfies (1) (resp. (2); resp. (3)). We say that *uniqueness in law* holds for (1) (resp. (2); resp. (3)), if any two weak solutions (X, W) and $(\tilde{X}, \tilde{W})$ have the same laws whenever the laws at the initial time $t = 0$ are the same.

Finally we say that *pathwise uniqueness* holds for equation (1) (resp. (2); resp. (3)) if two weak solutions (X, W) and $(\tilde{X}, W)$ defined on the same filtered probability space with the same Brownian motion W are such that the processes X and $\tilde{X}$ being undistinguishable. $\diamond$

There were many works which discuss under which conditions on the drift b and the diffusion coefficient σ we have the existence of strong solutions of stochastic differential equations. In the case when the equation is one-dimensional and σ is not degenerated, several results have been obtained by Y.Ouknine ([12],[13],[14]). For SDEs which involves the local time of the unknown process X, as far as we know, the most general result is given by M.Rutkowski in ([19]) where he showed that the so called **LT**-condition is sufficient to have pathwise uniqueness. In that case the diffusion coefficient σ can be degenerated. $\diamond$

2 Perturbed SDEs with measurable coefficients

To begin with, let us recall the following definition which we borrow from Ash & Doléans-Dade [1].

Definition 2.1. *A two-variable function $f(t, x)$ is called monotonically increasing if whenever $t_2 \geq t_1$ and $x_2 \geq x_1$ we have:*

$$f(t_2, x_2) - f(t_2, x_1) - f(t_1, x_2) + f(t_1, x_1) \geq 0.$$

To a monotonically increasing function $f(t, x)$ we associate the Lebesgue-Stieltjes measure defined, for any $t_2 \geq t_1$ and $x_2 \geq x_1$, by:

$$\mu([t_1, t_2) \times [x_1, x_2)) = f(t_2, x_2) - f(t_2, x_1) - f(t_1, x_2) + f(t_1, x_1).$$

Thus, for a measurable function $g(t,x)$, we can define its Lebesgue-Stieltjes integral w.r.t. f in the following way:

$$\int_{t_1}^{t_2}\int_a^b g(t,x)\,d_{t,x}f(t,x) = \int_{t_1}^{t_2}\int_a^b g(t,x)\,\mu(dt,dx).$$

Let us now make precise the assumptions on the coefficients b and σ:

(i) σ and b are measurable functions on $\mathbf{R}^+\times\mathbf{R}$ and sub-linearly growing, i.e.,

$$|\sigma(t,x)| + |b(t,x)| \le C\,(1+|x|)\,, \forall t\in\mathbf{R}^+ \text{ and } x\in\mathbf{R}. \tag{LG}$$

(ii) There exists a monotonically increasing function $f(t,x)$ on $\mathbf{R}^+\times\mathbf{R}$ such that $f(0,x)$ is increasing and

$$|\sigma(t,x)-\sigma(t,y)|^2 \le |f(t,x)-f(t,y)|\,, \forall t\in\mathbf{R}^+ \text{ and } x,y\in\mathbf{R}. \tag{BV2}$$

(iii) There exists $\varepsilon>0$ such that:

$$\sigma(t,x)\ge\varepsilon \text{ for any } t\in\mathbf{R}^+ \text{ and } x\in\mathbf{R}. \tag{ND}$$

Remark 2.1. *We can easily see that the mapping $x\mapsto f(t,x)$ is increasing for any fixed $t\in\mathbf{R}^+$.*

The main result of this section is the following

Theorem 2.2. *Let $\alpha\in(0,1)$. Then, under the conditions (LG), (BV2) and (ND) the perturbed SDE (1) has a strong solution which is moreover pathwise unique.*

Proof: It will be obtained after the following three steps.

Step 1: **Existence and uniqueness in law**

We start with a Brownian motion W, and consider the process $X_t = W_t + \frac{\alpha}{1-\alpha}\max_{0\le s\le t} W_s$. Then, X solves the SDE

$$X_t = W_t + \alpha\max_{0\le s\le t} X_s.$$

Therefore, using the condition (LG) and the Girsanov's theorem, we easily deduce the existence and uniqueness in law of the solution of the SDE

$$X_t = W_t + \int_0^t b(s,X_s)ds + \alpha\max_{0\le s\le t} X_s. \tag{5}$$

We now claim that uniqueness in law holds for solutions of (1). Actually since σ is bounded from below by a real positive constant, there is a one to one correspondence between the distributions of solutions of (1) and the ones of the same equation but with $\sigma\equiv 1$. Indeed, if Y is a solution to (1) and if we define T_t, for any $t\ge 0$, by

$$T_t = \inf\left\{s>0 : \int_0^s \sigma(u,Y_u)^2du \ge t\right\}$$

then the process $(Y_{T_t})_{t\geq 0}$ is a solution to (5), and vice versa.

Step 2: **Pathwise uniqueness**

Let us first show that $L^0_{\cdot}(X-Y)\equiv 0$, whenever X and Y denote any two solutions of the SDE (1) with the same underlying Brownian motion W. By the right continuity of $L^0_{\cdot}$ it is enough to prove that, for any $t\geq 0$,

$$\int_0^{+\infty}\frac{L_t^a\,(X-Y)}{a}da<+\infty.$$

Indeed, using the density occupation formula we can write for any $\delta>0$,

$$\int_\delta^{+\infty}\frac{L_t^a}{a}da=\int_0^t\frac{d\langle X-Y\rangle_s}{X_s-Y_s}1_{\{X_s-Y_s>\delta\}}=\int_0^t\frac{(\sigma(s,X_s)-\sigma(s,Y_s))^2}{X_s-Y_s}1_{\{X_s-Y_s>\delta\}}ds.$$

Applying the assumption (BV2) we obtain

$$\int_0^t\frac{(\sigma(s,X_s)-\sigma(s,Y_s))^2}{X_s-Y_s}1_{\{X_s-Y_s>\delta\}}ds\leq\int_0^t\frac{|f(s,X_s)-f(s,Y_s)|}{X_s-Y_s}1_{\{X_s-Y_s>\delta\}}ds.$$

As a consequence,

$$\mathbb{E}\left[\int_\delta^{+\infty}\frac{L_t^a(X-Y)}{a}da\right]\leq\ \mathbb{E}\left[\int_0^t\frac{|f(s,X_s)-f(s,Y_s)|}{X_s-Y_s}1_{\{X_s-Y_s>\delta\}}ds\right].\qquad(6)$$

Now, by a localization argument we may assume that f is bounded and we consider the sequence of functions f_n defined by

$$f_n(t,a)=(f(t,.)*\theta_n)(a)$$

where θ_n is the standard positive regularizing mollifiers sequence. So, for fixed $t\geq 0$,

$$f_n(t,a)\to f(t,a)\quad\text{for every } a\notin D_t,$$

where D_t is the denumerable set of discontinuous points of the function $a\mapsto f(t,a)$.

Hence, using successively Fatou's Lemma, the intermediate value theorem, the fact that $\sigma\geq\epsilon$ and

$$\frac{d}{dt}\langle\alpha X+(1-\alpha)Y\rangle_t=(\alpha\sigma(t,X_t)+(1-\alpha)\sigma(t,Y_t))^2\geq\epsilon^2$$

we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\frac{(f(s,X_s)-f(s,Y_s))}{X_s-Y_s}1_{\{X_s-Y_s>\delta\}}ds\right]\\
&\leq\liminf_{n\to+\infty}\mathbb{E}\left[\int_0^t\frac{(f_n(s,X_s)-f_n(s,Y_s))}{X_s-Y_s}1_{\{X_s-Y_s>\delta\}}ds\right]\\
&=\liminf_{n\to+\infty}\mathbb{E}\left[\int_0^t ds\int_0^1\frac{\partial f_n}{\partial a}(s,\alpha X_s+(1-\alpha)Y_s)d\alpha\right]\\
&=\liminf_{n\to+\infty}\int_0^1 d\alpha\mathbb{E}\left[\int_0^t\frac{\partial f_n}{\partial a}(s,\alpha X_s+(1-\alpha)Y_s)ds\right]\\
&\leq\frac{1}{\epsilon^2}\liminf_{n\to+\infty}\int_0^1 d\alpha\int_R da\mathbb{E}\left[\int_0^t\frac{\partial f_n}{\partial a}(s,a)dL_s^a(\alpha X+(1-\alpha)Y)\right].
\end{aligned}$$

Note that we have used in the first inequality the fact that

$$\int_0^t P[(X_s \in D_s) \cup (Y_s \in D_s)] ds \quad = \quad 0. \tag{7}$$

To see that this statement holds, it suffices to consider the case where $\sigma \equiv 1$ and $b \equiv 0$, the general situation may be deduced by applying both Girsanov theorem and a time change. In this case, as it was observed in the beginning of the proof, the solution may be expressed as a function of (W_t, M_t^W) , and since the law of the later process is absolutely continuous w.r.t the Lebesgue measure and D_s is denumerable we get (7).

Next, since f is monotonically increasing, the positivity of the measure $\dfrac{\partial^2 f_n}{\partial t \partial a}(dt, da)$ yields

$$\begin{aligned}
&\mathbb{E}\left[\int_R da \int_0^1 d\alpha \int_0^t \frac{\partial f_n}{\partial a}(s,a) dL_s^a(\alpha X + (1-\alpha)Y)\right] \\
&= \mathbb{E}\left[\int_R \int_0^1 \frac{\partial f_n}{\partial a}(t,a) L_t^a(\alpha X + (1-\alpha)Y) d\alpha da \right.\\
&\qquad \left. - \int_R \int_0^1 d\alpha \int_0^t \frac{\partial^2 f_n}{\partial t \partial a}(ds, da) L_s^a(\alpha X + (1-\alpha)Y)\right] \\
&\leq \mathbb{E}\left[\int_R \int_0^1 \frac{\partial f_n}{\partial a}(t,a) L_t^a(\alpha X + (1-\alpha)Y) d\alpha da\right].
\end{aligned}$$

Thus,

$$\mathbb{E}\left[\int_\delta^{+\infty} \frac{L_t^a(X-Y)}{a} da\right] \quad \leq \quad \liminf_{n \to +\infty} \mathbb{E}\left[\int_R \int_0^1 \frac{\partial f_n}{\partial a}(t,a) L_t^a(\alpha X + (1-\alpha)Y) d\alpha da\right]. \tag{8}$$

However, since $\alpha \in (0,1)$, then standard calculations imply that, for any $p \geq 0$ and $t \geq 0$,

$$\mathbb{E}[\sup_{s \leq t} |X_s|^p] < \infty. \tag{9}$$

So, if X^α is the process defined by $X^\alpha := \alpha X + (1-\alpha)Y$ we deduce by using Tanaka formula and the inequality $|X_t^\alpha - a| - |X_0^\alpha - a| \leq |X_t^\alpha - X_0^\alpha|$ that

$$\sup_{\alpha \in [0,1], a \in R} \mathbb{E}\left[L_t^a(\alpha X + (1-\alpha)Y)\right] < \infty.$$

Therefore, we obtain

$$\mathbb{E}\left[\int_\delta^{+\infty} \frac{L_t^a}{a} da\right] \quad \leq \quad \sup_{\alpha \in [0,1], a \in R} \mathbb{E}\left[L_t^a(\alpha X + (1-\alpha)Y)\right] \int_R \frac{\partial f_n}{\partial a}(t,a) da \leq C \left\|f\right\|_\infty \tag{10}$$

where $C > 0$ is a generic constant. The result follows by letting δ to zero in (10).

Step 3: **Extrema of two solutions are solutions**

We now show that $X \wedge Y$ and $X \vee Y$ are also solutions to (1). Using Tanaka's formula we write

$$\begin{aligned} X_t \vee Y_t &= Y_t + (X_t - Y_t)^+ \\ &= Y_t + \left(\int_0^t 1_{\{X_s > Y_s\}} d\left(X_s - Y_s\right) + \tfrac{1}{2} L_t^0 \left(X - Y\right) \right) \\ &= \int_0^t 1_{\{X_s > Y_s\}} dX_s + \int_0^t 1_{\{X_s \le Y_s\}} dY_s \\ &= \int_0^t \sigma(s, X_s \vee Y_s) dW_s + \int_0^t b(s, X_s \vee Y_s) ds \\ &\qquad + \alpha \left(\int_0^t 1_{\{X_s > Y_s\}} dM_s^X + \int_0^t 1_{\{X_s \le Y_s\}} dM_s^Y \right), \end{aligned} \tag{11}$$

where M_s^Z denotes $\max_{0 \le s \le t} Z_s$.

Next we claim that

$$M_t^{X \vee Y} = \int_0^t 1_{\{X_s > Y_s\}} dM_s^X + \int_0^t 1_{\{X_s \le Y_s\}} dM_s^Y, \forall t \ge 0.$$

Actually by commuting X and Y in (11) we obtain:

$$\int_0^t 1_{\{X_s = Y_s\}} dM_s^X = \int_0^t 1_{\{X_s = Y_s\}} dM_s^Y, \ \forall t \ge 0,$$

and then

$$\begin{aligned} M_t^{X \vee Y} &= \int_0^t 1_{\{X_s > Y_s\}} dM_s^{X \vee Y} + \int_0^t 1_{\{X_s \le Y_s\}} dM_s^{X \vee Y} \\ &= \int_0^t 1_{\{X_s > Y_s\}} dM_s^X + \int_0^t 1_{\{X_s < Y_s\}} dM_s^Y + \int_0^t 1_{\{X_s = Y_s\}} dM_s^{X \vee Y}. \end{aligned}$$

But

$$\begin{aligned} \int_0^t 1_{\{X_s = Y_s\}} d(M_s^{X \vee Y} - M_s^Y) &= \int_0^t 1_{\{X_s = Y_s\}} d(M_s^X - M_s^Y)^+ \\ &= \int_0^t 1_{\{M_s^X > M_s^Y\}} 1_{\{X_s = Y_s\}} d(M_s^X - M_s^Y) = 0. \end{aligned}$$

Thus

$$M_t^{X \vee Y} = \int_0^t 1_{\{X_s > Y_s\}} dM_s^X + \int_0^t 1_{\{X_s \le Y_s\}} dM_s^Y, \forall t \ge 0.$$

This shows that $X \vee Y$ is actually a solution for (1). Similarly one can show that $X \wedge Y$ is a solution as well. Now, as X and Y have integrable paths on finite intervals and taking into account the uniqueness in law we obtain:

$$\mathbb{E}[|X - Y|] = \mathbb{E}[X \vee Y - X \wedge Y] = 0,$$

therefore $X = Y$.

By the same lines as the previous proof we have the following more general setting:

Proposition 2.1. *Assume the existence and uniqueness in law for (1) holds and assume also that for any solutions X and Y defined on the same stochastic basis with respect to the same Brownian motion one has $L_\cdot^0(X - Y) = 0$, then (1) has a strong solution which is pathwise unique.*

3 Perturbed SDEs with reflecting boundary

In this section we focus on the stochastic differential equation (2), i.e.,

$$\begin{cases} X_t &= \int_0^t \sigma(s,X_s)dW_s + \int_0^t b(s,X_s)ds + \alpha \max_{0\le s\le t} X_s \ + L_t^0, \\ X_t &\ge 0, \ \forall t\ge 0, \end{cases}$$

where $(L_t^0)_{t\ge 0}$ is the local time of the process X in 0. To begin with, let us consider a particular case of the previous equation, namely,

$$\begin{cases} X_t &= W_t + \alpha \max_{0\le s\le t} X_s \ + L_t^0, \\ X_t &\ge 0, \ \forall t\ge 0. \end{cases} \tag{12}$$

Lemma 3.1. *There exists a unique solution in law for equation (12).*

Proof: If we set $Y_t = (X_t)^2$ then Y_t satisfies the following SDE:

$$Y_t = 2\int_0^t \sqrt{Y_s}dW_s + t + \alpha \max_{0\le s\le t} Y_s.$$

Actually this is due to the fact that for any $t \ge 0$, $\{\frac{Y_t}{\max_{0\le s\le t} Y_s}\}^{\frac12} d(\max_{0\le s\le t} Y_s) = d(\max_{0\le s\le t} Y_s)$. It follows, from a result by R.A. Doney, J. Warren and M. Yor [6] that this latter equation has a unique solution in law, and then X is unique in law.

We now consider the following SDE with reflecting boundary and a measurable drift b, i.e.,

$$\begin{cases} X_t &= W_t + \int_0^t b(s,X_s)ds + \alpha \max_{0\le s\le t} X_s \ + L_t^0, \\ X_t &\ge 0, \forall t\ge 0. \end{cases} \tag{13}$$

We then have the following result:

Proposition 3.2. *There exists a weak solution of (13) which is moreover unique in law.*

Proof: Let $(\bar{X}_t, \bar{W})_{t\ge 0}$ be a solution in law for (12) on a filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\ge 0}, \bar{P})$. Using Itô's formula with $(\bar{X})^p$ for $p \ge 2$ we deduce that for any $t \ge 0$, $\mathbb{E}[\sup_{s\le t} |\bar{X}_s|^p] < \infty$. This property combined with the fact b is of linear growth imply that for any $t \ge 0$ we have :

$$\bar{\mathbb{E}}\left[\exp\{-\int_0^t b(s,\bar{X}_s)d\bar{W}_s - \int_0^t b^2(s,\bar{X}_s)ds\}\right] = 1.$$

One can see e.g. the appendix of [8] for the proof. Then there exists a probability Q on $\bar{\Omega}$ such that the process $\tilde{W} := \{\bar{W}_t + \int_0^t b(s,\bar{X}_s)ds\}_{t\ge 0}$ is a Q-Brownian motion on $\bar{\Omega}$. It follows that the process $(\bar{X}_t)_{t\ge 0}$ satisfies the following SDE:

$$\begin{cases} \bar{X}_t &= \tilde{W}_t + \int_0^t b(s,\bar{X}_s)ds + \alpha \max_{0\le s\le t} \bar{X}_s \ + L_t^0 \\ \bar{X}_t &\ge 0, \forall\, t\ge 0. \end{cases}$$

Finally uniqueness in law follows from Girsanov's theorem.

Were are now ready to give the main result of this section.

Theorem 3.3. *For any $\alpha \in (0,1)$ the following equation:*

$$\begin{cases} X_t = \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds + \alpha \max_{0 \le s \le t} X_s \ + L_t^0(X), \\ X_t \ge 0, \forall t \ge 0. \end{cases} \tag{14}$$

has a unique strong solution which has pathwise uniqueness property

Proof: The existence/uniqueness in law is a consequence of Proposition 3.2 and the condition [ND] on σ, therefore we only prove pathwise uniqueness.

So suppose that X and Y are two solutions for the SDE (14) defined on a common probability basis with respect to the same Brownian motion W. The assumptions [ND] and [BV2] on σ imply that $L^0_{\cdot}(X - Y) = 0$ (see e.g. [17]).

We now show that $X \wedge Y$ and $X \vee Y$ are also solutions to (14). By Tanaka's formula we have:

$$\begin{aligned} X_t \vee Y_t &= Y_t + (X_t - Y_t)^+ \\ &= Y_t + \textstyle\int_0^t 1_{\{X_s > Y_s\}} d(X_s - Y_s) + \frac{1}{2} L_t^0(X - Y) \\ &= \textstyle\int_0^t 1_{\{X_s > Y_s\}} dX_s + \int_0^t 1_{\{X_s \le Y_s\}} dY_s \\ &= \textstyle\int_0^t \sigma(s, X_s \vee Y_s) dW_s + \int_0^t b(s, X_s \vee Y_s) ds \\ &\qquad + \alpha \left(\textstyle\int_0^t 1_{\{X_s > Y_s\}} dM_s^X + \int_0^t 1_{\{X_s \le Y_s\}} dM_s^Y\right) \\ &\qquad + \textstyle\int_0^t 1_{\{X_s > Y_s\}} dL_s^0(X) + \int_0^t 1_{\{X_s \le Y_s\}} dL_s^0(Y), \end{aligned}$$

where M_t^Z denotes $\max_{0 \le s \le t} Z_s$. Now from a result by Ouknine [11] (or, Ouknine & Rutkowski [15]), we have:

$$\int_0^t 1_{\{X_s > Y_s\}} dL_s^0(X) + \int_0^t 1_{\{X_s \le Y_s\}} dL_s^0(Y) = L_t^0(X \vee Y).$$

But,

$$M_t^{X \vee Y} = \int_0^t 1_{\{X_s > Y_s\}} dM_s^X + \int_0^t 1_{\{X_s \le Y_s\}} dM_s^Y.$$

Consequently, we get that $X \vee Y$ is actually a solution to (14). Similarly we can show that $X \wedge Y$is a solution as well. As X and Y have integrable paths on finite time interval, then:

$$E[|X - Y|] = E[X \vee Y - X \wedge Y] = 0$$

which implies that $X = Y$ whence pathwise uniqueness. $\diamond$

4 SDE involving maximum process in the drift

In this section we give an existence and pathwise uniqueness result for SDEs where the drift is a function of the maximum process. Those equations appear in recent result by

B. Roynette P. Vallois and M. Yor [18] see also J. Obloj and M. Yor [10] about non Markovian process satisfying the $\mathbf{2M}-\mathbf{X}$ property.

More precisely, let us consider the following SDE:

$$X_t = x + W_t + \int_0^t b(X_s, \max_{0\le s\le t} X_s)ds, \tag{15}$$

where b is measurable and bounded function on $\mathbf{R}\times\mathbf{R}$ to $\mathbf{R}$. We have the following result:

Theorem 4.1. *Assume that for each $x\in\mathbf{R}$, $y\to b(x,y)$ is strictly increasing on the set $\{y\in\mathbf{R}/y\ge x\}$. Then (15) has a strong solution which is pathwise unique.*

Proof: The existence and uniqueness in law is a consequence of Girsanov's theorem. Suppose now that X and Y are two solutions for SDE (15) defined on the same probability basis with respect to the same Brownian motion W, then $X-Y$ is continuous with bounded variation and thus $L^0_{\cdot}(X-Y)=0$.

Next applying Tanaka's formula yields:

$$\begin{aligned} X_t\vee Y_t &= Y_t+(X_t-Y_t)^+ \\ &= Y_t+\int_0^t 1_{\{X_s>Y_s\}}d(X_s-Y_s)+\tfrac{1}{2}L^0_t(X-Y) \\ &= x+\int_0^t 1_{\{X_s>Y_s\}}dX_s+\int_0^t 1_{\{X_s\le Y_s\}}dY_s \\ &= x+W_t+\int_0^t 1_{\{X_s>Y_s\}}b(X_s,M^X_s)ds \\ &\qquad +\int_0^t 1_{\{X_s\le Y_s\}}b(Y_s,M^Y_s)ds. \end{aligned} \tag{16}$$

Observe that by a symmetry argument we have: $\forall t\ge 0$,

$$\int_0^t 1_{\{X_s=Y_s\}}b(Y_s,M^Y_s)ds=\int_0^t 1_{\{X_s=Y_s\}}b(X_s,M^X_s)ds.$$

We will now prove that if $X_s>Y_s$ then $M^X_s\ge M^Y_s$. Actually let us set

$$s_0=\sup\{v\in[0,s],\, X_v=Y_v\}.$$

Since X_s-Y_s is absolutely continuous then $\dfrac{d}{ds}(X_s-Y_s)_{s=s_0}\ge 0$ and thus

$$b(X_{s_0},M^X_{s_0})\ge b(Y_{s_0},M^Y_{s_0}).$$

Since $(b(x,y))_{y\ge x}$ is strictly increasing and through the definition of s_0 we deduce that $M^X_{s_0}\ge M^Y_{s_0}$. Furthermore for $v\in[s_0,s]$ we have $X_v>Y_v$ and then $M^X_s\ge M^Y_s$ which is the desired result.

Going back to (16) we obtain:

$$\begin{aligned} X_t\vee Y_t &= x+W_t+\int_0^t 1_{\{X_s>Y_s\}}b(X_s,M^X_s)ds+\int_0^t 1_{\{X_s\le Y_s\}}b(Y_s,M^Y_s)ds \\ &= x+W_t+\int_0^t 1_{\{X_s>Y_s\}}b(X_s,M^X_s)ds+\int_0^t 1_{\{X_s<Y_s\}}b(Y_s,M^Y_s)ds \\ &\qquad +\int_0^t 1_{\{X_s=Y_s\}}b(Y_s,M^Y_s)ds \\ &= x+W_t+\int_0^t 1_{\{X_s>Y_s\}}b(X_s\vee Y_s,M^{X\vee Y}_s)ds \\ &\qquad +\int_0^t 1_{\{X_s<Y_s\}}b(X_s\vee Y_s,M^{X\vee Y}_s)ds+\int_0^t 1_{\{X_s=Y_s\}}b(Y_s,M^Y_s)ds \\ &= x+W_t+\int_0^t b(X_s\vee Y_s,M^{X\vee Y}_s)ds-\int_0^t 1_{\{X_s=Y_s\}}b(Y_s,M^{X\vee Y}_s)ds \\ &\qquad +\int_0^t 1_{\{X_s=Y_s\}}b(Y_s,M^Y_s)ds. \end{aligned}$$

But

$$\begin{array}{rcl} \int_0^t 1_{\{X_s=Y_s\}} b(Y_s, M_s^Y)ds - \int_0^t 1_{\{X_s=Y_s\}} b(X_s, M_s^{X\vee Y})ds & = & \\ \int_0^t 1_{\{X_s=Y_s\}\cap\{M_s^X > M_s^Y\}} \left(b(Y_s, M_s^Y) - b(X_s, M_s^X)\right) ds & = & 0. \end{array}$$

This proves that $X \vee Y$ is also a solution for equation (15). Now uniqueness in law implies that pathwise uniqueness holds, whence the desired result. $\diamond$

5 Some possible extensions

5.1 The perturbed SDE (1) may be regarded as a particular case of the the following doubly perturbed SDE

$$X_t = \xi + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds + \alpha \max_{0\le s\le t} X_s + \beta \min_{0\le s\le t} X_s, \tag{17}$$

where $\alpha, \beta \in \mathbb{R}$.
As it was mentioned in the introduction, this doubly perturbed SDE was the subject of studies of several authors (see the reference in the beginning of the introduction). Carmona, Petit and Yor ([2]) shows, in the particular case when $b \equiv 0$ and $\sigma \equiv 1$, the existence and pathwise uniqueness of the strong solution by using a fixed point argument, under the following conditions on the parameters α and β :

$$\begin{cases} \alpha < 1, \ \beta < 1 \\ \dfrac{|\alpha\beta|}{(1-\alpha)(1-\beta)} < 1. \end{cases} \tag{18}$$

Here we get strong existence and pathwise uniquness of the solution to (17), in the Lipschitz case by using the Picard iteration. More precisely, if σ and β are taken to satisfy:

$$\begin{cases} |\sigma(s,x) - \sigma(s,y)| & \le c|x-y| \\ |b(x) - b(y)| & \le c|x-y| \end{cases} \tag{19}$$

for every x, y in $\mathbb{R}, s \ge 0$ and some constant $c > 0$, then we have :

Theorem 5.1. *Under the assumptions (18) and (19), and if the random variable ξ is such that $\mathbb{E}(|\xi|^2) < \infty$, then there exists a unique continuous $\mathcal{F}^W$ adapted process $(X_t, t \ge 0)$ which is a solution to the doubly perturbed SDE (17). Moreover $\mathbb{E}(\max_{0\le s\le T} |X_s|^2) < \infty$ for every $T > 0$.*

Proof: *Proof of the existence*: The construction of the solution uses the Picard iteration. Let us consider the sequence implicitly defined by:

$$\begin{cases} X_t^0 = \dfrac{\xi}{1-\alpha}, \\ X_t^{n+1} = \xi + \displaystyle\int_0^t \sigma(X_s^n)dW_s + \int_0^t b(X_s^n)ds + \alpha M_t^{n+1} + \beta I_t^{n+1}, \end{cases} \tag{20}$$

where $M_t^{n+1} \triangleq \max_{0\le s\le t} X_s^{n+1}$ and $I_t^{n+1} \triangleq \min_{0\le s\le t} X_s^{n+1}$ for every $t \ge 0$.
Observe that $(X^n)_{n\ge 0}$ is well defined. In fact X^{n+1} is explicitly evaluated from X^n, since by Skorohod's lemma [17], one can easy seen that:

$$\begin{cases} (1-\alpha)M_t^{n+1} &= \max_{0\le s\le t}\left(\xi + \int_0^s \sigma(X_u^n)dW_u + \int_0^s b(X_u^n)du + \beta I_s^{n+1}\right), \\ (\beta-1)I_t^{n+1} &= \max_{0\le s\le t}\left(-\xi - \int_0^s \sigma(X_u^n)dW_u - \int_0^s b(X_u^n)du - \alpha M_s^{n+1}\right). \end{cases} \tag{21}$$

Combining these two equalities, we get:

$$\begin{aligned} M_t^{n+1} \quad &= \quad \frac{1}{1-\alpha}\max_{0\le s\le t}\left\{\left(\xi + \int_0^s \sigma(X_u^n)dW_u + \int_0^s b(X_u^n)du\right)\right. \\ &+ \quad \left.\frac{\beta}{\beta-1}\max_{0\le u\le s}\left(-\xi - \int_0^u \sigma(X_v^n)dW_v - \int_0^u b(X_v^n)dv - \alpha M_u^{n+1}\right)\right\}. \end{aligned} \tag{22}$$

By a similar argument as in [2], using a fixed point theorem and the hypothesis (18), we get the well definiteness and the adaptation of M_t^{n+1} with respect to the filtration of B. Thus, $(X^n)_{n\ge 0}$ is well defined.
Now, let us show that X^n converges uniformly on compact intervals almost surely. On one hand, we have:

$$\max_{0\le s\le t}|X_s^{n+1} - X_s^n| \le a_n(t) + b_n(t) + |\alpha|\max_{0\le s\le t}|M_s^{n+1} - M_s^n| + |\beta|\max_{0\le s\le t}|I_s^{n+1} - I_s^n|. \tag{23}$$

On the other hand, by (21)

$$\begin{cases} \max_{0\le s\le t}|M_s^{n+1} - M_s^n| \le \frac{1}{1-\alpha}\{a_n(t)+b_n(t)\} + \frac{|\beta|}{1-\alpha}\max_{0\le s\le t}|I_s^{n+1} - I_s^n|, \\ \max_{0\le s\le t}|I_s^{n+1} - I_s^n| \le \frac{1}{1-\beta}\{a_n(t)+b_n(t)\} + \frac{|\alpha|}{1-\beta}\max_{0\le s\le t}|M_s^{n+1} - M_s^n|, \end{cases} \tag{24}$$

where $a_n(t) := \max_{0\le u\le t}|\int_0^u (\sigma(X_s^n) - \sigma(X_s^{n-1}))dW_s)|$ and $b_n(t) := \int_0^t |b(X_s^n) - b(X_s^{n-1})|ds$, for $t \ge 0$. Thus,

$$\left(1 - \frac{|\alpha\beta|}{(1-\alpha)(1-\beta)}\right)\max_{0\le s\le t}|M_s^{n+1} - M_s^n| \le \frac{1}{1-\alpha}\left(1 + \frac{|\beta|}{1-\beta}\right)(a_n(t)+b_n(t)). \tag{25}$$

Combining (23) and (25) yields

$$\begin{aligned} \max_{0\le s\le t}|X_s^{n+1} - X_s^n| \quad &\le \quad \left(1 + \frac{|\beta|}{1-\beta}\right)\left(1 + \frac{|\alpha|(1-\beta)}{(1-\alpha)(1-\beta) - |\alpha\beta|}\right. \\ &+ \quad \left.\frac{|\alpha\beta|}{(1-\alpha)(1-\beta) - |\alpha\beta|}\right)(a_n(t)+b_n(t)). \end{aligned}$$

Applying Bukholder-Davis-Gundy's inequality and Lipshitz's conditions to this inequality, we obtain for a generic constant $c > 0$,

$$\mathbb{E}\left(\max_{0\le s\le t}|X_s^{n+1} - X_s^n|^2\right) \le c\int_0^t \mathbb{E}|X_s^n - X_s^{n-1}|^2 ds. \tag{26}$$

Iterating this inequality led, for every $T > 0$, to

$$\mathbb{E}\left[\max_{0\le s\le t}|X_s^{n+1} - X_s^n|^2\right] \le c\frac{T^n}{n!} \tag{27}$$

since $\mathbb{E}\xi^2 < \infty$. Consequently, by the Chebychev's inequality and the Borel-Cantelli lemma, we get the uniform convergence of the sequence $(X^n)_n$ to a continuous process X on $[0,T]$(see for instant [9]). Letting $n \to \infty$ in (20) it follow that X is a solution to the doubly perturbed SDE (17). As T is arbitrary this proves the existence.
Proof of the uniqueness: Now suppose that X and Y are two solution to the SDE (17) with some initial condition ξ and some driving Brownian motion B, then we can easily seen

$$|X_t - Y_t| \le |\int_0^t(\sigma(X_s) - \sigma(Y_s))dW_s| + |\beta||I_t^X - I_t^Y| + \frac{|\alpha|}{1-\alpha}\max_{0\le s\le t}|(A_s^X)^+ - (A_s^Y)^+| \tag{28}$$

with $A_t^\omega := \xi + \int_0^t \sigma(s,\omega_s)dB_s + \int_0^t b(s,\omega_s)ds + \alpha I_t^X$ where ω is X or Y.
Arguing as above (using Skorohod's lemma), we have

$$\left(1 - \frac{|\alpha\beta|}{(1-\alpha)(1-\beta)}\right)\max_{0\le s\le t}|I_s^X - I_s^Y| \le c(a(t) + b(t)), \tag{29}$$

where $a(t) := \max_{0\le u\le t}|\int_0^u(\sigma(X_s) - \sigma(Y_s))dW_s)|$ and $b(t) := \int_0^t |b(X_s) - b(Y_s)|ds$.
Moreover,

$$|A_t^X - A_t^Y| \le |\int_0^t(\sigma(X_s) - \sigma(Y_s))dW_s| + \int_0^t |b(X_s) - b(Y_s)|ds + |\beta||I_t^X - I_t^Y|$$

which implies, taking into account of (29)

$$\max_{0\le s\le t}|A_s^X - A_s^Y| \le c(a(t) + b(t)) \tag{30}$$

and consequently

$$|X_t - Y_t| \le c(a(t) + b(t)). \tag{31}$$

Finally, by applying BDG's inequality and the lipshitz condition we get

$$\mathbb{E}(|X_t - Y_t|^2) \le c\int_0^t \mathbb{E}\left(|X_u - Y_u|^2|\right)du.$$

Hence, $\mathbb{E}\left(|X_t - Y_t|^2\right) = 0$ by Gronwall's lemma. Thus the solution is unique. $\diamond$

Remark 5.1. *It would be interesting to see, if the result of Theorem.5.1 can be proved by the approach used in the first section of this paper. This will illuminate the situation when the coefficients σ and b are not Lipschitz.*

5.2 Let H be an absolutely continuous and increasing function such that $0 < H'(x) < 1$ and $H(0) = 0$. We can develop the same arguments as previously to show that our results on strong existence and pathwise uniqueness are still valid for the following SDEs:

$$X_t = \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds + H\left(\max_{0\le s\le t} X_s\right),\ \forall t \ge 0 \tag{32}$$

or

$$\begin{cases} X_t &= X_0 + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds + H\left(\max_{0\le s\le t} X_s\right) + L_t^0 \\ X_t &\ge 0,\ \forall t \ge 0 \end{cases} \tag{33}$$

and finally

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds + \int_0^t \alpha(s)dM_s^X\ \forall t \ge 0, \tag{34}$$

for any deterministic Borel function α such that $0 \le \alpha(s) < 1$. $\diamond$

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